DOCUMENT RESUME

ED 469 611 SE 066 413

AUTHOR	Beem, Paul
TITLE	Spiral Symmetry on the TI-92.
PUB DATE	1998-07-00
NOTE	9p.; Paper presented at the International DERIVE/TI-92 Conference (3rd, Gettysburg, PA, July 14-17, 1998).
PUB TYPE	Guides - Non-Classroom (055) -- Speeches/Meeting Papers (150)
EDRS PRICE	EDRS Price MF01 Plus Postage. PC Not Available from EDRS
DESCRIPTORS	Computer Uses in Education; Curriculum Development; *Geometric Concepts; Geometry; *Graphing Calculators; Mathematics Activities; *Mathematics Instruction; Secondary Education

ABSTRACT

This paper introduces a teaching unit on using spiral symmetry in a geometry course for secondary education majors. The TI-92 graphing calculator is used to exploit the exponential function to analyze the geometric topic of similarity and to study the derivative of spirals. The paper provides some procedures and programs to generate spirals using the TI-92 graphing calculator. (KHR)

Reproductions supplied by EDRS are the best that can be made
from the original document.

U.S. DEPARTMENT OF EDUCATION
Office of Educational Research and Improvement
EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)
This document has been reproduced as received from the person or organization originating it.
Minor changes have been made to improve reproduction quality.
Points of view or opinions stated in this document do not necessarily represent official OERI position or policy.

Proceedings of the Third International DERIVE/TI-92 Conference

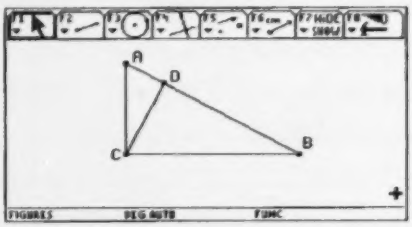
ED 469 611

Spiral Symmetry on the TI-92

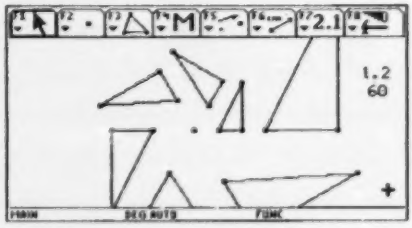
Paul Beem
Indiana University South Bend
South Bend, IN
pbeem@iusb.edu

We all think and reason in context. In a linear algebra class, we try to use linear equations and homomorphisms and, when in a geometry class, lines and circles. The same student who easily finds the exponential function as the answer to a radioactive decay problem in a calculus class won't necessarily find it as the solution to a similarity problem in a geometry class. This talk is about a unit I teach on spiral symmetry in my Topics In Geometry course for secondary education majors. Spiral symmetry exploits the exponential function to analyze the geometric topic of similarity. The TI-92 is used throughout.

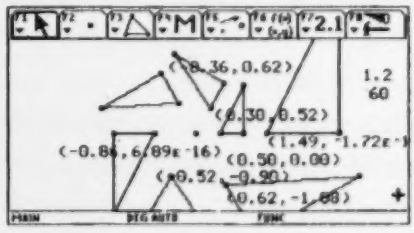
We start with the notion of a dilative rotation. This is a dilation similarity (a stretching or shrinking) centered at a point P followed by a rotation around the same point. The amount of stretching (or shrinking) is called the dilation number and may be negative. (Two dilative rotations are considered the same if their dilation numbers are opposites and their angles differ by 180 degrees.) Suppose ΔABC has a right angle at C and a perpendicular is dropped from C to the hypotenuse $\overline{AB}$ meeting $\overline{AB}$ at D. There is a dilative rotation centered at D that maps ΔACD to ΔCBD. The angle of the transformation is +90 degrees and the dilation number is the ratio BC/AC. Dilative rotations are sometimes called spiral symmetries and we will now see why.



Suppose we start with a certain triangle and perform the same dilative rotation on it and its iterates a number of times. We get, of course, a series of triangles, each rotated by the same angle from the previous one and expanded (or contracted) by the same ratio. (In the figure, the ratio is 1.2 and the angle is 60 degrees. Suppose we keep track of a particular vertex of this triangle and record its various positions as it rotates around the center of the dilative rotation. It is clear that these points lie on a spiral of some sort. We will use the TI-92 to make this precise.



Suppose we pick the vertex at the larger acute angle, on the original triangle, and construct the coordinates of it and its iterates. We arrange these coordinates so that they separated sufficiently to be selected easily. Then we put



PERMISSION TO REPRODUCE AND DISSEMINATE THIS MATERIAL IN MICROFICHE, AND IN ELECTRONIC MEDIA FOR ERIC COLLECTION SUBSCRIBERS ONLY, HAS BEEN GRANTED BY

R.P. Beem

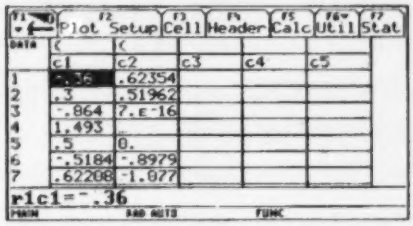
TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)

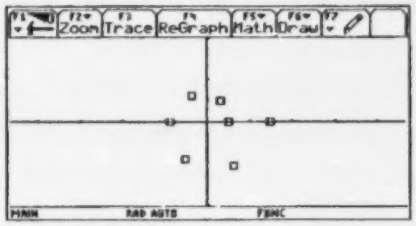
2A

ED 469 613

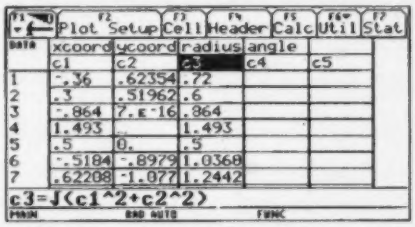
BEST COPY AVAILABLE

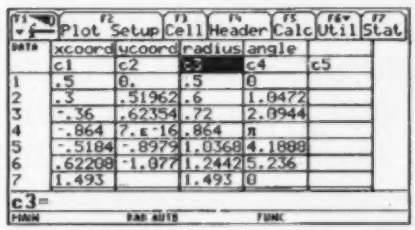
these data into the data base variable SYSDATA. Because of the way the selection procedure works, it is important to select the x-coordinate first and to deselect both coordinates before selecting the next pair. Now plot this data and we see the plot of points we collected in the previous step.

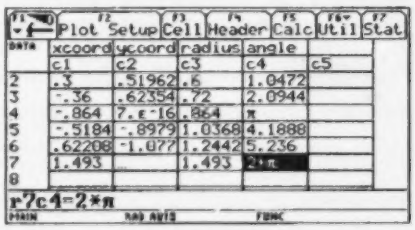




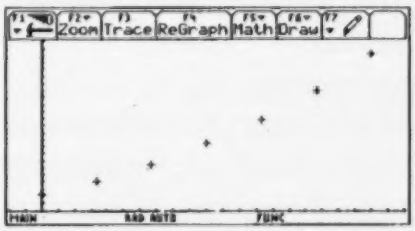
To make sense of this data, it is convenient to convert it into polar coordinates. Returning to SYSDATA, we label the first two columns, XCOORD and YCOORD. We label the next two columns RADIUS and ANGLE and define C3 to be SQRT(C1^2+C2^2). If we were to use the built-in inverse tangent function, we would get values in the range $-\pi/2$ to $\pi/2$, which is not what we want. But it is easy to define a custom Arctan function using the built-in ANGLE function. Using this function and adding 2π, "by hand", to account for the winding nature of the data, we get it in usable form.







Next, we plot the radius against the angle. Make sure that the calculator is in FUNCTION mode and that angles are measured in radians, not degrees. Because of the way the points were constructed, we know that their second coordinates are increasing by a constant ratio for every regular change in the angle. (In the example, the modulus of the point is increasing by 1.2 for every 60° change in the angle.) This type of function is usually called exponential growth.

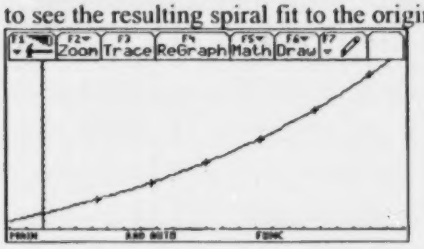


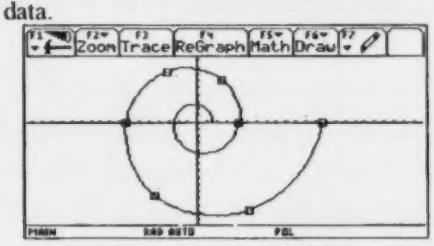
We can check our guess that this is an exponential function by doing an exponential regression on the data. We can view the resulting exponential function (the regression

BEST COPY AVAILABLE

Contents

function) in both FUNCTION mode, in order to see the usual graph, or in POLAR form, to see the resulting spiral fit to the original data.





We can derive the equation of the spiral easily. First note that, if λ and μ denote the constant angle and dilation number, respectively, then the coordinate (r,θ) transforms, via the dilative rotation, to $(r\cdot\mu, \theta+\lambda)$. Since the dilative rotation is a symmetry of the curve, $r(\theta+\lambda) = \mu \cdot r(\theta)$. Setting $\theta = 0, \lambda, 2\lambda, \ldots$, we get $r(n\cdot\lambda) = \mu^n \cdot r(0)$. Let a denote $r(0)$ and $\theta = n\lambda$. Then $r(\theta) = a\mu^{\frac{\theta}{\lambda}}$. The simplest assumption to make now is to assume that θ can take on any value. This then becomes the equation of our "self similar" curve. (In the example, $r(\theta) = 0.5(1.2)^{\frac{30}{\pi}\theta}$.)

Let $D(\rho,\xi)$ denote the dilative rotation centered at the origin with dilation number ρ and rotation angle ξ.

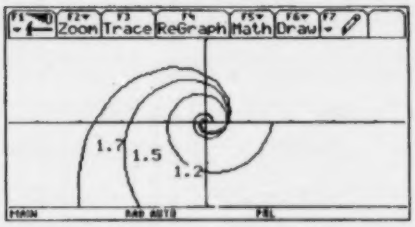
Theorem: The spiral $r(\theta) = a\mu^{\frac{\theta}{\lambda}}$ has $D(\rho,\xi)$ as a symmetry if and only if $\frac{\ln(\mu)}{\lambda} = \frac{\ln(\rho)}{\xi}$.

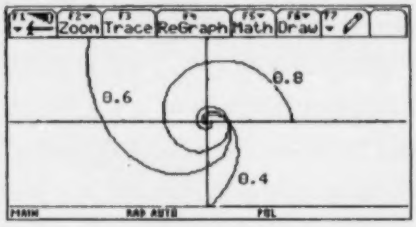
Proof: $D(\rho,\xi)$ is a symmetry of the curve $r(\theta) = a\mu^{\frac{\theta}{\lambda}}$ if and only if $r(\theta+\xi) = \rho\cdot r(\theta)$. This is equivalent to $a\mu^{\frac{\theta+\xi}{\lambda}} = \rho(a\mu^{\frac{\theta}{\lambda}})$. Dividing both sides by $a\mu^{\frac{\theta}{\lambda}}$ and taking logarithms yields the result.

Corollary: The above spiral can also be written as $r(\theta) = a\mu_0^{\ \theta}$, where μ_0 denotes the dilation number required to stay on the spiral through a rotation of one radian.

Proof: Just take $\xi = 1$, in the theorem.

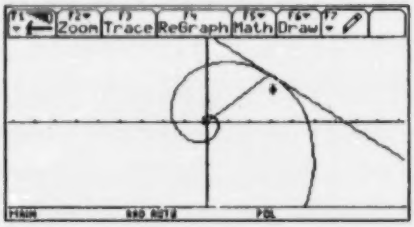
The number μ_0 is called the natural dilation number of the spiral. If $\mu_0 > 1$, the spiral winds out counter-clockwise and if $0 < \mu_0 < 1$, then the spiral winds out clockwise.





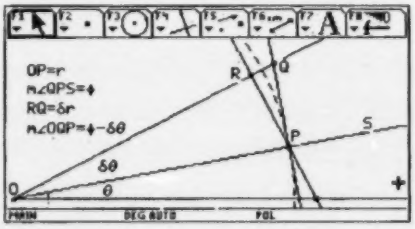
We can also express the spiral in terms of the base e: $r(\theta) = ae^{\theta \ln(\mu_0)}$. The quantity $\ln(\mu_0)$ has a geometric interpretation.

Theorem: The equation of the spiral can be written in the form $r(\theta) = ae^{\theta \cot(\phi)}$, where $\phi = \cot^{-1}(\ln(\mu_0))$ is the angle between the radius vector to a point on the spiral and the tangent vector to the spiral at the same point.



Note, in particular, that the angle ϕ is a constant, and is not dependent on θ. Logarithmic spirals are also called equiangular spirals because of this property. Note also that we can assume that ϕ is a first or second quadrant angle, depending on whether the spiral winds out counter-clockwise or clockwise. ϕ

Proof: Let O denote the origin and P a point on the spiral $r(\theta) = a\mu_0^{\theta}$ with (polar) coordinates (r,θ). Increment the angle θ by a small amount $\delta\theta$ and let Q be the intersection of the tangent line at P and the augmented radius vector (see figure.) Let R be the foot of the perpendicular from P to the line OQ.

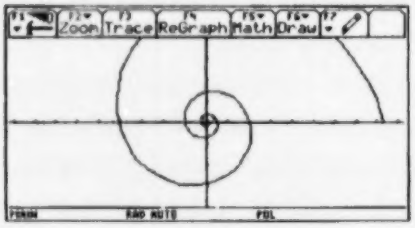


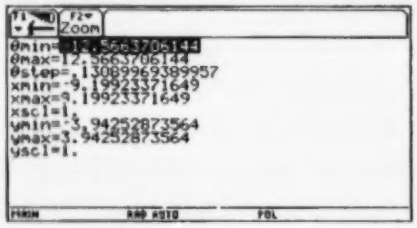
Then $OP = r$, $m\angle QPS = \phi$, $RQ \approx \delta r$, and $m\angle OQP = \phi - \delta\theta$. Let $\eta = m\angle OQP$. Then $RP = r \cdot \sin(\delta\theta)$. Since $\sin(\delta\theta) \approx \delta\theta$, when $\delta\theta$ is small, $\cot(\eta) = RQ/RP \approx \dfrac{\delta r}{r\delta\theta}$, when $\delta\theta$ is small.

Taking the limit on both sides of $cot(\phi - \delta\theta) \approx \frac{\delta r}{r\delta\theta}$, as $\delta\theta$ goes to zero, yields

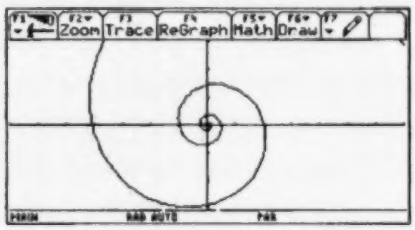
$$\cot(\phi) = \frac{1}{r}\frac{dr}{d\theta} = \ln(\mu_0).$$

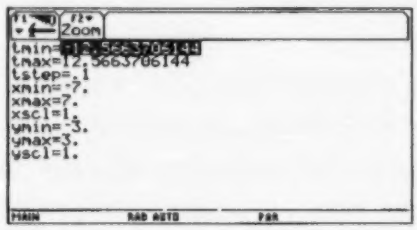
Next, we will use the calculator to study the derivative spiral. We start by defining $r1(\theta) = (0.5)(.8)^{\theta}$ and graphing it, in polar mode, using a square window.



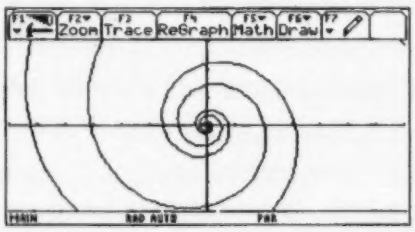


Now, on the home screen, define $xt1(t) = r1(t)\cos(t)$ and $yt1(t) = r1(t)\sin(t)$. Graph these equations in parametric mode, again with a square screen.





Now return to the home screen and define $xt2(t) = \frac{d}{dt}(xt1(t))$ and $yt2(t) = \frac{d}{dt}(yt1(t))$. Now graph both the spiral and its derivative. (The graphing will go faster if you obtain explicit formulas for the right hand sides of these expressions before defining the left hand sides.) If you graph these in simultaneous mode with their styles set to PATH, you will see an interesting relationship between these two curves.



Now return to the home screen and simplify the expressions for the derivatives. If you factor and then tCollect each expression, you should get

$$\frac{d}{dt}(xt1(t)) = -0.5122...(.8)^{t}\sin(t + 0.2195...)$$

BEST COPY AVAILABLE

$$\frac{d}{dt}(ytl(t)) = +0.5122...(.8)^t \sin(t + 1.7903...).$$

Who are these strange numbers and what do they want? First of all, note that $\frac{\pi}{2} + (t + 0.2195...) = t + 1.7903...$, so that the equations can be written

$$\frac{d}{dt}(xtl(t)) = -0.5122...(.8)^t \cos(t + 1.7903...)$$
$$\frac{d}{dt}(ytl(t)) = +0.5122...(.8)^t \sin(t + 1.7903...)$$

which is a little better. Next note that $\cot(1.7903...) = \ln(.8)$, so that 1.7903 is the constant angle ϕ.

To get better control over this, we return to some theory. Since the spiral has the equations:

$$x(t) = a\mu_0{}^t \cos(t)$$
$$y(t) = a\mu_0{}^t \sin(t)$$

the derivatives will have the form:

$$x'(t) = a(\ln(\mu_0)\cos(t) - \sin(t))\mu_0{}^t = \ln(\mu_0)x(t) - y(t)$$
$$y'(t) = a(\cos(t) + \ln(\mu_0)\sin(t))\mu_0{}^t = x(t) + \ln(\mu_0)y(t).$$

In matrix form, these equations become:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} \ln(\mu_0) & -1 \\ 1 & \ln(\mu_0) \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

Factoring out the square root of the determinant (why not?) yields:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = D \begin{bmatrix} \frac{\ln(\mu_0)}{D} & \frac{-1}{D} \\ \frac{1}{D} & \frac{\ln(\mu_0)}{D} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \text{ where } D = \sqrt{1 + \ln^2(\mu_0)}.$$

The matrix is now in the form of a rotation matrix, the rotation being by an angle θ_0, where $\sin(\theta_0) = \frac{1}{D}$ and $\cos(\theta_0) = \frac{\ln(\mu_0)}{D}$. It follows that $\cot(\theta_0) = \ln(\mu_0)$, so that $\theta_0 = \phi$, the constant angle for the spiral. Note that, in this form, θ_0 is a first or second quadrant angle depending on whether $\mu_0 \geq 1$ or $\mu_0 \leq 1$.

So, we have shown that the derivative curve is a dilative rotation of the original spiral. The dilation number is $D = \sqrt{1 + \ln^2(\mu_0)} = \csc(\phi)$ and the angle is ϕ, where ϕ denotes the constant angle for the original spiral. We can check this against our example above. In our case, $r1(\theta) = (0.5)(.8)^{\theta}$, so that μ_0 is .8. Hence, $\ln(\mu_0) = -.2231\ldots$ and $D = 1.0245\ldots$. Hence θ_0 is a second quadrant angle and is given by $\pi - \sin^{-1}(\frac{1}{D}) = 1.79034\ldots$. This checks with the above. The product of D and a is $(.5)(1.0245\ldots) = 0.5122\ldots$, which also checks.

It is not hard to see that the curve $(x'(t), y'(t))$ is also a logarithmic spiral, since:

$$x'(t) = Da(\cos(t)\cos(\theta_0) - \sin(t)\sin(\theta_0))\mu_0{}^t = Dr(t)\cos(t+\theta_0) = D\mu_0{}^{-\theta_0} r(t+\theta_0)\cos(t+\theta_0)$$

$$y'(t) = Da(\cos(t)\sin(\theta_0) + \sin(t)\cos(\theta_0))\mu_0{}^t = Dr(t)\sin(t+\theta_0) = D\mu_0{}^{-\theta_0} r(t+\theta_0)\sin(t+\theta_0)$$

Hence, the point $(x'(t), y'(t))$ lies on the curve $r(\theta) = (D\mu_0{}^{-\theta_0} a)\mu_0{}^{\theta}$, a spiral parallel (i.e., having the same natural base) to the original spiral.

We have seen, in the above, that the derivative of a logarithmic spiral is another logrithmic spiral and is a dilation of the original spiral. It is not hard to show that any dilation of a logarithmic spiral is another logarithmic spiral. In fact, the same is true of any rotated logrithmic spiral.

Theorem: Let **C** be the graph of the logarithmic spiral $r(\theta) = a\mu^{\theta}$. Then both $D(\nu,0)(\mathbf{C})$ and $D(1,\lambda)(\mathbf{C})$ are graphs of logarithmic spirals and they are the same graph if $\nu = \mu^{-\lambda}$.

Proof: Suppose (r,θ) maps to the point (r',θ'') via the dilation $D(\nu,0)$. Then $D(\frac{1}{\nu},0)(r',\theta') = (r,\theta) = (a\mu^{\theta},\theta)$. Hence, $\frac{1}{\nu} r' = a\mu^{-\theta}$. Since $\theta = \theta'$, the dilated spiral has equation $r(\theta') = (a\nu)\mu^{\theta'}$, which is a spiral parallel to the original spiral.

Suppose, on the other hand that (r,θ) maps to (r',θ'') via the rotation $D(1,\lambda)$. Then $r' = r$ and $\theta' = \theta + \lambda$. Since $r(\theta) = a\mu^{\theta}$ on **C**, $r' = a\mu^{\theta'-\lambda} = (a\mu^{-\lambda})\mu^{\theta'}$. That is, the image of $D(1,\lambda)(\mathbf{C})$ of **C** has polar equation $r(\theta) = (a\mu^{-\lambda})\mu^{\theta}$, which again is a spiral parallel to the original spiral.

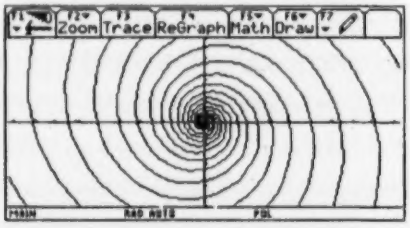
Equating the two expressions for the radius vector, yields the result $\nu = \mu^{-\lambda}$.

In particular then, the above theorem says that dilating the spiral $r(\theta) = a\mu^{\theta}$ by a factor $\mu^{-\lambda}$ has the same effect as rotating the spiral by λ. This can be illustrated on the calculator. Change to polar graphing mode and set a small square window with a symmetric θ range. Then type in the following program:

```
rotspirl()
Prgm
Local i
For i,1,6,1
θmin-(i-1)*π/3→ θmin
θmax-(i-1) )*π/3 → θmax
0.5*(0.8)^( θ+(I-1) )*π/3) → r1(θ)
DispG
StoPic #("spir"&string(i))
EndFor
EndPrgm
```

This program creates and stores six pictures (.PIC) files each of which requires 3097 bytes for a total of 18582 bytes of memory. If you don't have that available, either delete some unneccessary files or increase the increment angle (which is $\pi/3$ in the program) and make fewer pictures.

The picures will be stored as SPIR1.PIC, SPIR2.PIC, … , SPIR6.PIC. You can view the pictures using the VAR-LINK viewing function or by opening them in the graph screen. The following picture shows all six spirals:



To see even more vividly the equivalence between dilations and rotations of this spiral, write the command CYCLEPIC "SPIR", 6, .1, 10, -1. The resulting animation seems to be of a spiral dilating in and out, even though it was written as a rotating spiral.

BEST COPY AVAILABLE

. . .

EdD913.

U.S. Department of Education
Office of Educational Research and Improvement (OERI)
National Library of Education (NLE)
Educational Resources Information Center (ERIC)

REPRODUCTION RELEASE

(Specific Document)

I. DOCUMENT IDENTIFICATION:

Title: Steiner's Porism: An Activity Using the TI-92 and Spiral Symmetry on the TI-92

Author(s): Paul Beem

Corporate Source:	Publication Date:

II. REPRODUCTION RELEASE:

In order to disseminate as widely as possible timely and significant materials of interest to the educational community, documents announced in the monthly abstract journal of the ERIC system, *Resources in Education* (RIE), are usually made available to users in microfiche, reproduced paper copy, and electronic media, and sold through the ERIC Document Reproduction Service (EDRS). Credit is given to the source of each document, and, if reproduction release is granted, one of the following notices is affixed to the document.

If permission is granted to reproduce and disseminate the identified document, please CHECK ONE of the following three options and sign at the bottom of the page.

The sample sticker shown below will be affixed to all Level 1 documents	The sample sticker shown below will be affixed to all Level 2A documents	The sample sticker shown below will be affixed to all Level 2B documents
PERMISSION TO REPRODUCE AND DISSEMINATE THIS MATERIAL HAS BEEN GRANTED BY TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC) 1	PERMISSION TO REPRODUCE AND DISSEMINATE THIS MATERIAL IN MICROFICHE, AND IN ELECTRONIC MEDIA FOR ERIC COLLECTION SUBSCRIBERS ONLY, HAS BEEN GRANTED BY R Paul Beem TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC) 2A	PERMISSION TO REPRODUCE AND DISSEMINATE THIS MATERIAL IN MICROFICHE ONLY HAS BEEN GRANTED BY TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC) 2B
Level 1 ↑	Level 2A ↑	Level 2B ↑
Check here for Level 1 release, permitting reproduction and dissemination in microfiche or other ERIC archival media (e.g., electronic) and paper copy.	Check here for Level 2A release, permitting reproduction and dissemination in microfiche and in electronic media for ERIC archival collection subscribers only	Check here for Level 2B release, permitting reproduction and dissemination in microfiche only

Documents will be processed as indicated provided reproduction quality permits.
If permission to reproduce is granted, but no box is checked, documents will be processed at Level 1.

I hereby grant to the Educational Resources Information Center (ERIC) nonexclusive permission to reproduce and disseminate this document as indicated above. Reproduction from the ERIC microfiche or electronic media by persons other than ERIC employees and its system contractors requires permission from the copyright holder. Exception is made for non-profit reproduction by libraries and other service agencies to satisfy information needs of educators in response to discrete inquiries.

Sign here, please

Signature: R. Paul Beem	Printed Name/Position/Title: R. Paul Beem Assoc. Prof. of MATH	
Organization/Address: Indiana University South Bend	Telephone: 574-237-4375	FAX:
	E-Mail Address: pbeem@iusb	Date: 10/17/02

III. DOCUMENT AVAILABILITY INFORMATION (FROM NON-ERIC SOURCE):

If permission to reproduce is not granted to ERIC, or, if you wish ERIC to cite the availability of the document from another source, please provide the following information regarding the availability of the document. (ERIC will not announce a document unless it is publicly available, and a dependable source can be specified. Contributors should also be aware that ERIC selection criteria are significantly more stringent for documents that cannot be made available through EDRS.)

Publisher/Distributor:
Address:
Price:

IV. REFERRAL OF ERIC TO COPYRIGHT/REPRODUCTION RIGHTS HOLDER:

If the right to grant this reproduction release is held by someone other than the addressee, please provide the appropriate name and address:

Name:
Address:

V. WHERE TO SEND THIS FORM:

Send this form to the following ERIC Clearinghouse:

However, if solicited by the ERIC Facility, or if making an unsolicited contribution to ERIC, return this form (and the document being contributed) to:

ERIC Processing and Reference Facility
4483-A Forbes Boulevard
Lanham, Maryland 20706

Telephone: 301-552-4200
Toll Free: 800-799-3742
FAX: 301-552-4700
e-mail: info@ericfac.piccard.csc.com
WWW: http://ericfacility.org